Variational calculus problems part 2 - The brachistochrone-problem

An article posted on bathtubphysics.de

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2023-05-07

This article tackles two topics. Minimizing a path and a travel time inside a gravitational field.

The brachistochrone (from Greek, meaning the path of shortest travel time) is the result for the latter and will be discussed in more detail.

Consider a homogenious gravitational field acting with constant force "down". (So, this might be a region close to earth's surface, where the force due to gravity is approximately constant.)

Furthermore, consider two points $A(x_A, y_A)$ and $B(x_B, y_B)$, which are connected by a track. A pearl is placed at A with initial speed $v_0 = 0$ and will follow the track. (Friction will be ignored.) How must the track be shaped, so that the pearl will travel from A to B? (B shall be below A.) In particular:

What track shape would make the pearl reach B

- **following the shortest path?**
- **in the shortest time?**

The shortest path between two points - a straight line

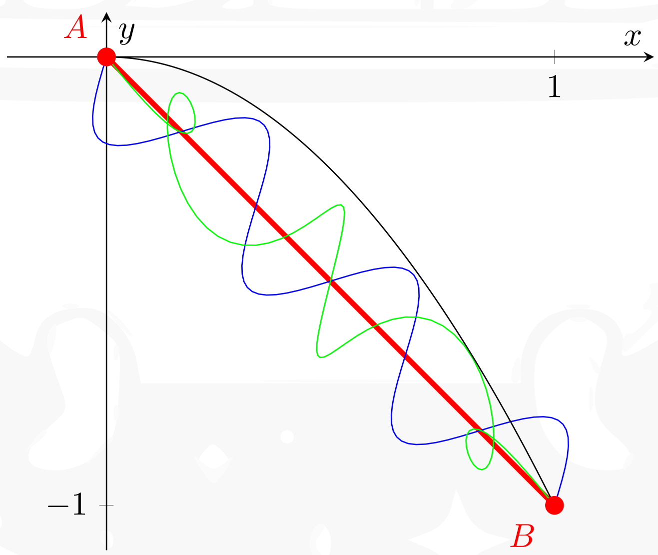

Figure 1: Some possible paths between two points. Obviosly, the thick red straight line is the shortest one.

In order to find the shortest path between A and B, we can use the general expression for the arc length $\mathcal{B}_{A,B}[y(x), y'(x)]$ of a given curve $y(x)$. (Formally, the arc length is a functional.)

$$\mathcal{B}_{A,B} = \int\limits_{x_a}^{x_b} \sqrt{1 + (y'(x))^2} dx$$

Then, the special $y(x)$ must be found that minimizes $\mathcal{B}_{A,B}$. Intuitively - and also actually - this will be a straight line. (So this part can be considered an introductory example.)

Now, how can we find the special function $y(x)$ that minimizes the functional $\mathcal{B}_{A,B}$?

We can use the Euler-Lagrange-equation

$$\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'} = \frac{\partial \mathcal{L}}{\partial y},$$

where $\mathcal{L}$, the so-called Lagrange-function (or more accurately, the Lagrange-functional).

In this case,

$$\mathcal{L}[y(x), y'(x), x] = \sqrt{1 + (y'(x))^2}.$$

In general, the Euler-Lagrange-equation yields “candidates” for functions $y(x)$ that minimize or maximize the functional

$$\mathcal{S}[y(x), y'(x), x] = \int_{x_a}^{x_b} \mathcal{L}[y(x), y'(x), x] \, dx.$$

At the time of writing this article, I am also writing a series on the whole Lagrange formalism.

The reader might find additional insights there.

In our case, the “candidate” will be an actual solution, which we will not prove.

First of all, the Lagrange-functional does not explicitly depend on y, which makes the right-hand-side of the Lagrange-equation just 0:

$$\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'} = 0$$

This immediately implies

$$\frac{\partial \mathcal{L}}{\partial y'} = \text{const.} =: k,$$

which leads to

$$\begin{aligned} k =& \frac{\partial}{\partial y'} \sqrt{1 + [y'(x)]^2} \\ =& \frac{y'}{\sqrt{1 + [y'(x)]^2}} \\ \Rightarrow \quad y' =& k\sqrt{1 + [y'(x)]^2} \\ \Rightarrow \quad (y')^2 =& k^2 + k^2 (y')^2 \\ \Rightarrow \quad k^2 =& (1 - k^2)(y')^2 \\ \Rightarrow \quad y' =& \pm \underbrace{\sqrt{\frac{k^2}{1 - k^2}}}_{=:m} . \end{aligned}$$

Notice, m is still a constant!

Thus,

$$\begin{aligned} y' =& m \\ \frac{dy}{dx} =& m \\ \Rightarrow \quad dy =& m \, dx \\ \Rightarrow \quad \int dy =& \int m \, dx \end{aligned}$$

$$\boxed{\Rightarrow \quad y = mx + c}$$

The solution is a straight line, indeed. As stated, we have not proven that this really is the function **minimizing** the arc length. After all, it could also maximize the arc length or be something different. (Even if this is obviously not the case, here.)

What's more, it is conceivable that the true solution to a problem might not be representable as a simple function $y(x)$. (Just picture a problem whose solution would be a circle.)

It is possible to handle these details as well, but here, we will settle for the plausible result paired with physical intuition, as this was - as stated above - merely an introductory example.

Path of minimal travel time - the brachistochrone

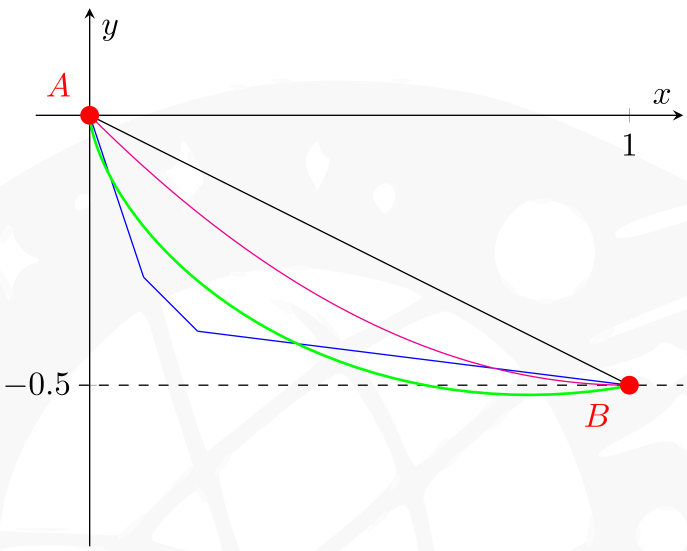

Figure 2: Some possible tracks between two points. Placing a pearl at the top left (red) point A, which track would get the pearl to the lower right point B the fastest?

Now, we want to find the fastest path from A to B instead of the shortest. The curve satisfying this is called "brachistochrone". (We ignore friction.)

By comparing the example curves in the plot above, we can think about possible effects that might have an influence.

- The black straight line is the shortest path. The acceleration of the pearl due to gravity is constant and overall, comparatively low.
- The blue curve has quite a long arc length, but it is also quite steep in the beginning, so the pearl quickly gets accelerated to a high velocity.
- The green curve is sort of a compromise between the above two curves. It still has a fairly long arc length, it is not as steep in the beginning but in comparison to the blue curve, it is steeper for longer. It is also the only one going **below** the destination point, giving it the highest top speed.
- The magenta curve (which happens to be a squished parabola) is, again, like another compromise between the green curve and the straight line.

Using functionals and the Euler-Lagrange-equation, we can already put numbers to this problem. However, it is useful to consider the so-called "Legendre-transformation" $\mathcal{H}$ of the Lagrange-functional $\mathcal{L}$, since it will drastically simplify solving the equations we will encounter. It is given by:

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial y'} y' - \mathcal{L}$$

Now, to handle our current setup, we place A right into the origin of our coordinate system and B inside the lower right quadrant, i. e. $y_B < 0$ and $x_B > 0$. Now, if we put a pearl on the track at the origin, it will move “down right”.

The travel time T can be expressed as:

$$\begin{aligned} T &= \int_0^T dt \\ &= \int_0^T \frac{ds}{ds} dt \\ &= \int_0^{\mathcal{B}} \frac{ds}{\frac{ds}{dt}} \end{aligned}$$

ds is an infinitesimal part of the curve’s arc length in such a way, that integrating over all ds will result in the full arc length.

$$\mathcal{B} = \int ds.$$

Now, $\frac{ds}{dt}$ becomes the pearl’s current speed v at a given point of the curve.

Since we assumed the pearl’s initial velocity to be $v(0) = 0$, we can denote the kinetic energy in its usual form of

$$E_{kin} = \frac{1}{2} m v^2.$$

From the law of conservation of energy, gain in kinetic Energy must equal loss in potential energy.

Therefore,

$$\begin{aligned} E_{kin} &= -E_{pot} \\ \Rightarrow \quad \frac{1}{2} m v^2 &= -ymg. \end{aligned}$$

Notice, that y (i. e. the height coordinate of the pearl at time t) is negative because of how the coordinate frame was chosen. (Thus, both sides of the equation will be positive and increasing over time.)

This implies:

$$\boxed{v = \sqrt{-2gy}}$$

Considering

$$ds = \sqrt{1 + (y')^2}\, dx,$$

the travel time functional T becomes

$$T = \int_0^{x_B} \frac{\sqrt{1 + (y')^2}}{\sqrt{-2gy}}\, dx.$$

That's something we know how to handle since the part inside the integral must satisfy the Euler-Lagrange-equation:

$$\mathcal{L} = \frac{\sqrt{1 + (y')^2}}{\sqrt{-2gy}}, \qquad \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'} = \frac{\partial \mathcal{L}}{\partial y}$$

As $\mathcal{L}$ does not explicitly depend on the x-coordinate, we can make use of the Legendre-transformation, which becomes a conserved quantity.

> This is quite easy to demonstrate:
>
> $$\begin{aligned} \frac{d}{dx}\mathcal{H} =& \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'} y' - \mathcal{L}\right) \\ =& \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'} y'\right) - \frac{d\mathcal{L}}{dx} \\ =& \left(\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'}\right) y' + \frac{\partial \mathcal{L}}{\partial y'} y'' - \frac{\partial \mathcal{L}}{\partial y} y' - \frac{\partial \mathcal{L}}{\partial y'} y'' - \frac{\partial \mathcal{L}}{\partial x} \\ =& \left[\left(\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial \mathcal{L}}{\partial y}\right] y' - \frac{\partial \mathcal{L}}{\partial x} \end{aligned}$$
>
> By applying the Euler-Lagrange-equation to the term inside the square brackets, we see that it vanishes. So, we end up with:

> $$\frac{d}{dx}\mathcal{H} = -\frac{\partial \mathcal{L}}{\partial x}$$
>
> Indeed, if the Lagrange-functional $\mathcal{L}$ does not explicitly depend on x, the Legendre-transformation $\mathcal{H}$ is a conserved quantity, i. e. a constant.

We can write

$$y'\frac{\partial \mathcal{L}}{\partial y'} - \mathcal{L} = \text{const.} =: -\hat{k}$$

or

$$\begin{aligned} -\hat{k} =& y' \cdot \left[\frac{\partial}{\partial y'}\frac{\sqrt{1+(y')^2}}{\sqrt{-2gy}}\right] - \frac{\sqrt{1+(y')^2}}{\sqrt{-2gy}} \\ \Leftrightarrow\ -\hat{k}\sqrt{-2gy} =& \left[\frac{(y')^2}{\sqrt{1+(y')^2}}\right] - \frac{1+(y')^2}{\sqrt{1+(y')^2}} \\ \Leftrightarrow\ \hat{k}\sqrt{-2gy} =& \frac{1}{\sqrt{1+(y')^2}}. \end{aligned}$$

Defining

$$k := \hat{k} \cdot \sqrt{2g}$$

as a new, not-yet-fixed constant, the upper equation can be rewritten in a slightly simpler form:

$$\begin{aligned} \sqrt{1+(y')^2} =& \frac{1}{k\sqrt{-y}} \\ \Rightarrow\ 1+(y')^2 =& \frac{1}{-k^2 y} \\ \Leftrightarrow\ (y')^2 =& \frac{1}{-k^2 y} - 1 \\ \Rightarrow\ y' =& \pm\sqrt{\frac{1}{-k^2 y} - 1} \end{aligned}$$

Now, which sign out of the $\pm$ do we choose?

Well, the slope of the track must be negative close to the point A (i. e. the coordinate frame's origin), or the pearl will not follow it. So, we choose the negative sign.

$$y' = -\sqrt{\frac{1}{-k^2 y} - 1}$$

Sadly, the solution of this differential equation cannot be represented in closed form. (Even if the inverse function $x(y)$ can be.) But we can use a very simple parameterized representation of the solution instead, which is even more physically intuitive. This can be done using these substitutions:

$$\begin{aligned} y(\tau) =& -R(1-\cos\tau) \\ x(\tau) =& R(\tau - \sin\tau) \end{aligned}$$

This parameterization can be thought of as the trace of a point on the outskirts of a circle-wheel that is rolling along the x-axis. (From below in this case.)

To visualize this effect, let us create plots for $R = 1$ and $\tau \in [0, 2\pi]$:

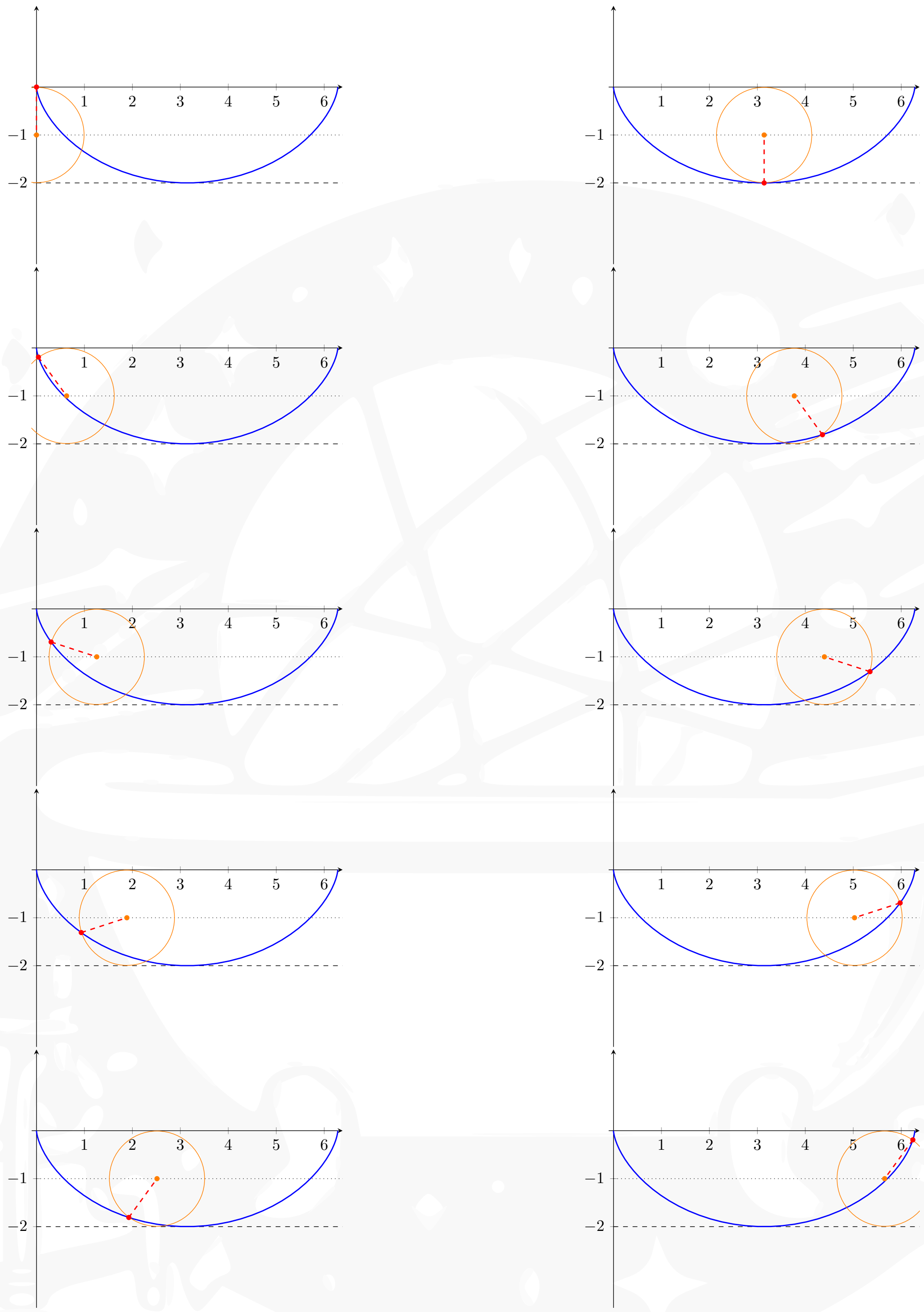

Figure 3: The brachistochrone constructed by tracing an outskirt point on a rolling wheel.

To demonstrate, the parameterizations satisfy the differential equation, we best use an "earlier" form of it:

$$\begin{aligned} \sqrt{-y}\sqrt{1+(y'(x))^2} =& \frac{1}{k} \\ \Rightarrow \ -y \cdot \left[1+(y'(x))^2\right] =& \text{const.} : c \end{aligned}$$

Remember, k was a real-number constant. Thus, we know $c = \frac{1}{k^2}$ is positive. Furthermore, y is limited to the interval $[-2R; 0]$ by our parameter representation, since the cosine has a limited range. This already gives us information on the curve's minimum at x_0, namely $y(x_0) = -2R$.

Getting back to the differential equation, using

$$\begin{aligned} y'(x) =& \frac{dy}{dx} \\ =& \frac{\partial y}{\partial \tau}\frac{\partial \tau}{\partial x} \\ =& \frac{\frac{\partial y}{\partial \tau}}{\frac{\partial x}{\partial \tau}} \\ =& \frac{-R\sin\tau}{R(1-\cos\tau)} \\ =& \frac{-\sin\tau}{1-\cos\tau}, \end{aligned}$$

y and y' can be inserted:

$$\begin{aligned} R(1-\cos\tau)\left[1+\frac{\sin^2\tau}{(1-\cos\tau)^2}\right] =& c \\ \Leftrightarrow \ R(1-\cos\tau)^2\left[1+\frac{\sin^2\tau}{(1-\cos\tau)^2}\right] =& c(1-\cos\tau) \\ \Leftrightarrow \ R\left[(1-\cos\tau)^2+\sin^2\tau\right] =& c(1-\cos\tau) \\ \Leftrightarrow \ R\left[1-2\cos\tau+\cos^2\tau+\sin^2\tau\right] =& c(1-\cos\tau) \\ \Leftrightarrow \ R\left[2-2\cos\tau\right] =& c(1-\cos\tau) \\ \Rightarrow \ 2R =& c \end{aligned}$$

In conclusion, if R is positive, it is always possible to choose c in such a way, that the parameter representation satisfies the differential equation. Now, we have to account for the boundary conditions as well. With

$$y(\tau) = R(\cos\tau - 1)$$
$$x(\tau) = R(\tau - \sin\tau)$$

representing the whole curve, we can insert the points A and B. But A being the coordinate origin, $\tau = 0$ will give us the right point for any value of R.

For any given point B, we have coordinates x_B and y_B and are left with two equations containing the two unknown variables R and τ.

They are not linear and thus, from a formal point of view, there is no guarantee that a unique solution actually always exists. But if nothing else, this might get us hopeful.

By dividing both equations, we obtain

$$\rho(\tau) := \frac{y_B}{x_B} = \frac{\cos\tau - 1}{\tau - \sin\tau},$$

which must be negative. (By the choice of the coordinate system. Remeber, that y_B is negative and x_B is positive by construction.) See the plot below.

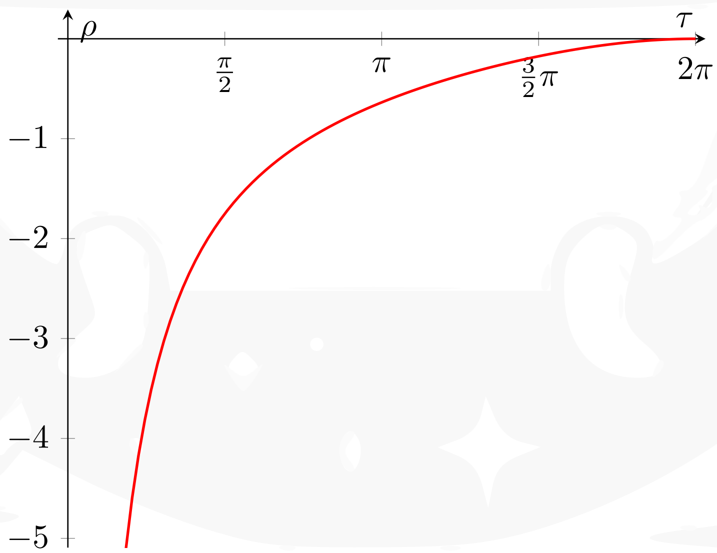

Figure 4: $\rho(\tau)$ on the interval from 0 to 2π.

The derivative of this function, $\rho'(\tau)$ is:

$$
\begin{aligned}
\rho'(\tau) =& \frac{d}{d\tau}\left[\frac{\cos\tau - 1}{\tau - \sin\tau}\right] \\
=& \frac{\frac{d}{d\tau}\left[\cos\tau - 1\right] \cdot (\tau - \sin\tau) - (\cos\tau - 1) \cdot \frac{d}{d\tau}\left[\tau - \sin\tau\right]}{(\tau - \sin\tau)^2} \\
=& \frac{-\sin\tau \cdot (\tau - \sin\tau) - (\cos\tau - 1) \cdot [1 - \cos\tau]}{(\tau - \sin\tau)^2} \\
=& \frac{-\tau\sin\tau + \sin^2\tau + (1 - \cos\tau)^2}{(\tau - \sin\tau)^2} \\
=& \frac{-\tau\sin\tau + \sin^2\tau + 1 - 2\cos\tau + \cos^2\tau}{(\tau - \sin\tau)^2} \\
=& \frac{-\tau\sin\tau + 2\,(1 - \cos\tau)}{(\tau - \sin\tau)^2}
\end{aligned}
$$

Particularly, we care for the sign of this function on the interval $(0, 2\pi]$.

We could try to do a lengthy formal proof, but here, we will just have a look at the plot.

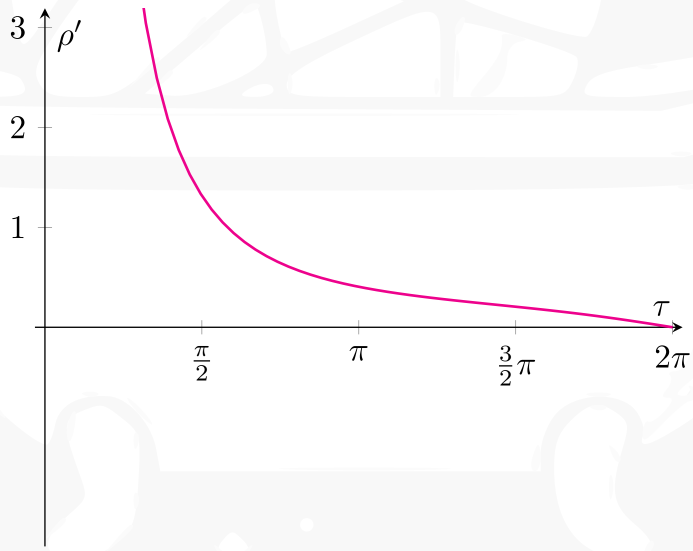

Figure 5: $\rho'(\tau)$ on the interval from 0 to 2π.

That seems to check out. The function $\rho(\tau)$ is monotonously increasing on $(0, 2\pi]$, because its derivative is positive on this interval.

Alternatively, it can be argued that if we consider the limit $\tau \to 0$ for the function $\rho(\tau)$, it goes to negative infinity. By inserting $\tau = 2\pi$, it is just equal to 0. Since the function is continuous on $x \in (0, 2\pi]$ by being a combination of continuous functions, every negative real number must be hit by at least one

$\tau \in (0, 2\pi]$.

However, the derivative being positive makes the original function monotonous, which guarantees that the function is also bijective, i. e. every possible ρ corresponds to *exactly one* τ.

Thus, for any given B, we can find a specific τ to achieve the proper ratio of coordinates. In turn, R becomes nothing more than a "scaling factor" for the whole curve.

The $R = 1$-plot that was already given is the general shape of the brachistochrone. It can only be scaled up or down, but it will always look like this.

The question really becomes, where to identify point B on this curve.

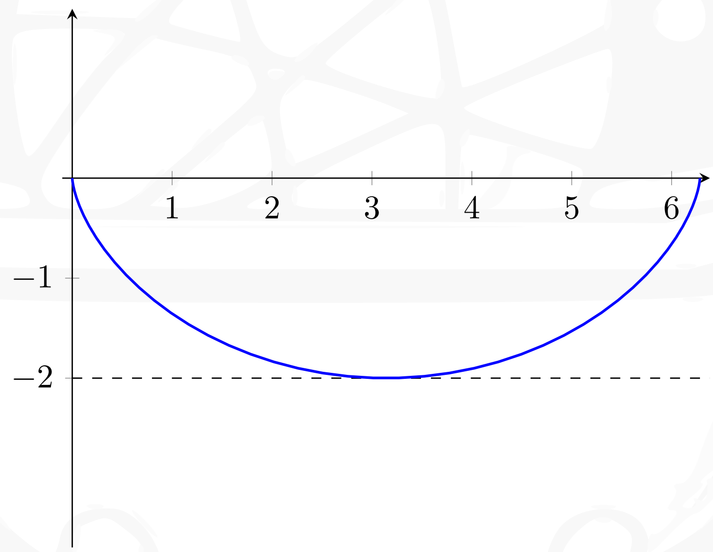

Figure 6: The normalized brachistochrone. ($R = 1$).

Since every τ uniquely corresponds to a certain $\rho = \frac{y_B}{x_B}$, we can always find a unique and unambiguous τ, albeit this will probably require numerical methods.

Notice, that $\rho = \tan\varphi$ can also be expressed using the angular coordinate φ in regular polar coordinates. After all, the polar angle φ's tangent is defined as this very ratio $\frac{y}{x}$. Using the polar angle instead of the ratio ρ, our plot becomes:

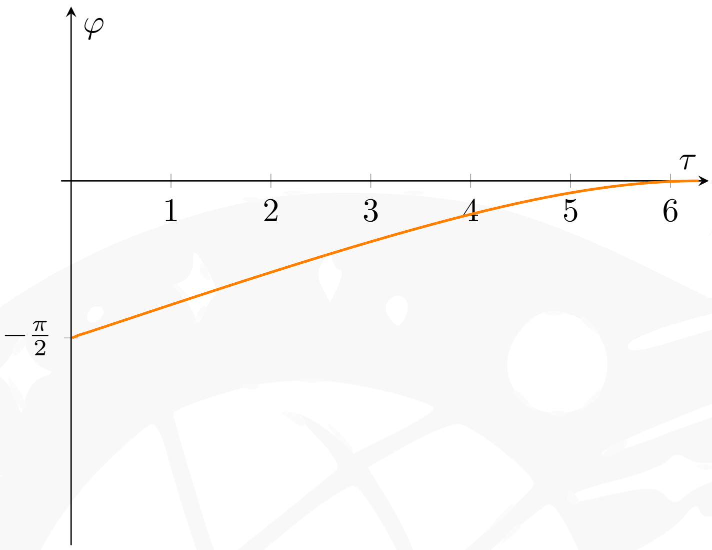

Figure 7: $\varphi(\tau)$. Every angle between $-\frac{\pi}{2}$ and 0 corresponds uniquely to a particular τ.

As we can see, for every $\varphi \in [-\frac{\pi}{2}, 0]$, we get one unique value of $\tau \in [0, 2\pi]$.

In other words, since $\varphi(\tau)$ is monotonous and continuous, every angle φ inside the fourth quadrant uniquely corresponds to a value of τ between 0 and 2π.

The constant parameter R can now be calculated from the ratio

$$R = \frac{y_B}{Y_B},$$

where

$$Y_B = \cos \tau_B - 1$$

is specified by the very τ_B, that was derived in the last step.

Using $x_B = 1 \;\Rightarrow\; \rho_B = y_B$ we can calculate values for some example cases:

ρ_B	τ_B	R
-10	~ 0.299	~ 225.385
-1	~ 2.412	~ 0.573
$-\frac{2}{\pi}$	π	$\frac{1}{\pi}$

ρ_B	τ_B	R
-0.5	~ 3.508	~ 0.259

Notice, that for the last example curve, the pearl will have to **ascend** towards the end of the track. This always happens, if x_B is to the right of the curve's minimum, i. e. for $\tau_B > \pi$.

The following plots illustrate the situation:

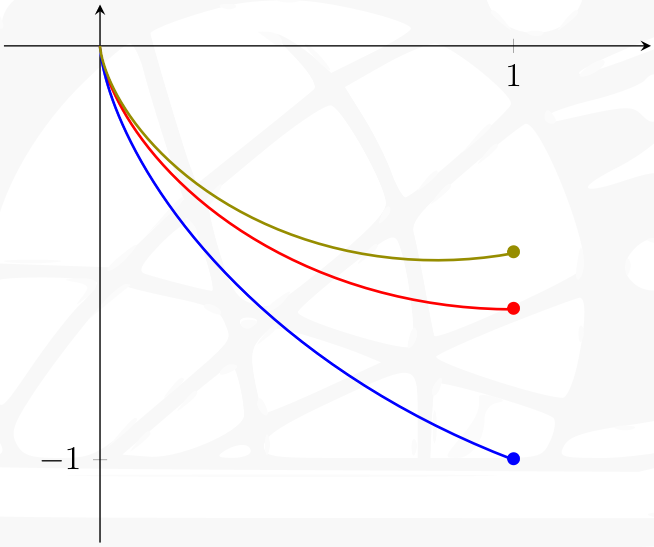

Figure 8: Three of the brachistochrones from the above tabular. They all have $x_B = 1$, while y_B-coordinates are -1, $-\frac{2}{\pi}$ and -0.5 in blue, red and olive, respectively.

As mentioned, the minimum can be found at $\tau_0 = \pi$. (At this point, $\cos\tau_0 - 1 = -2$ is minimal and in turn, so is $Y_0(\tau_0)$.) For ρ, we obtain

$$\rho(\tau_0) = \rho(\pi) = \frac{\cos\pi - 1}{\pi - \sin\pi} = -\frac{2}{\pi}.$$

Due to ρ' being positive on $\tau \in (0, 2\pi]$, we know, that for a larger τ, we get a larger ρ as well. In turn, so is y for a fixed x. If τ goes down, so does ρ and, again, y for a fixed x.

Therefore, a ratio of $\frac{y_B}{x_B}$ being larger than $-\frac{2}{\pi}$ means, that τ is larger than $\tau_0 = \pi$ and thus, we must be to the right of the minimum. (Where we need a final ascent to get there.)

In other words:

If

$$\rho = \frac{y_B}{x_B} > -\frac{2}{\pi},$$

B sits to the right of the minimum and the pearl will have to ascend at the end of the track.

The brachistochrone being a tautochrone

The brachistochrone has another interesting property. It is also the tautochrone curve, meaning that a pearl (with initial speed 0) will always require the same amount of time to reach the minimum, no matter what point on the track was its initial position.

This can be demonstrated by going back to the time functional.

We can start at any location $x_S < x_0$ and go up to the minimum x_0. The time to reach the minimum becomes:

$$T_{min}(x_S) = \int_{x_S}^{x_0} \frac{\sqrt{1+(y')^2}}{\sqrt{-2g(y-y_S)}}\, dx$$

Notice, that since our pearl will no longer start at point A, i. e. in the coordinate frame's origin, we need to slightly adjust the denominator. (Physically, we made the kinetic energy 0 in (x_S, y_S) instead of the origin.)

Recall, that in the denominator, we have an expression that represents the current velocity of the pearl, which depends on the "height" from which the pearl has already traveled down.

When the starting point had a y-coordinate of 0, this height was just $-y$, but here, it is $y_S - y$ or $-(y - y_S)$. Thus, we replaced accordingly.

This integral can be rewritten in terms of y. First, we consider

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{dx^2+dy^2} = -\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy.$$

We need the additional minus sign, because the way we had chosen the coordinate frame, dy is negative for a positive dx. (At least to the left of the minimum.)

With this, we get:

$$T_{min}(y_S) = \int\limits_{y_S}^{y_0} -\frac{\sqrt{1+(\frac{dx}{dy})^2}}{\sqrt{-2g(y-y_S)}}\, dy = \int\limits_{y_0}^{y_S} \frac{\sqrt{1+(\frac{dx}{dy})^2}}{\sqrt{-2g(y-y_S)}}\, dy$$

Now, let us use our parameter equations for the brachistochrone curve:

$$\begin{aligned} y(\tau) =& R(\cos\tau - 1) \\ x(\tau) =& R(\tau - \sin\tau) \end{aligned}$$

We can invert $y(\tau)$ to

$$\tau(y) = \arccos\left(\frac{y}{R}+1\right),$$

which we can insert into $x(\tau)$ to make x a function of y, too. In particular,

$$x(y) = R\left[\arccos\left(\frac{y}{R}+1\right) - \sin\left(\arccos\left(\frac{y}{R}+1\right)\right)\right].$$

With

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}},$$

the chain rule can be used to compute the derivative $\frac{dx}{dy}$:

$$\begin{aligned} \frac{dx}{dy} =& R\left[\frac{d}{dy}\arccos\left(\frac{y}{R}+1\right) - \frac{d}{dy}\left[\sin\left(\arccos\left(\frac{y}{R}+1\right)\right)\right]\right] \\ =& R\left[-\frac{1}{R}\sqrt{1-\left(\frac{y}{R}+1\right)^2}^{-1} - \cos\left(\arccos\left(\frac{y}{R}+1\right)\right)\frac{d}{dy}\arccos\left(\frac{y}{R}+1\right)\right] \\ =& \frac{y}{R}\sqrt{1-\left(\frac{y}{R}+1\right)^2}^{-1} \end{aligned}$$

Since we already know, that the minimum is located at $y_0 = -2R$, we can substitute $y = R\cdot\upsilon$ and use $\upsilon_0 = -2$:

$$\begin{aligned}
T_{min}(\upsilon_S) &= \int_{-2}^{\upsilon_S} \frac{\sqrt{1+\frac{\upsilon^2}{1-(\upsilon+1)^2}}}{\sqrt{-2gR(\upsilon-\upsilon_S)}} R d\upsilon \\
&= \sqrt{\frac{R}{g}} \int_{-2}^{\upsilon_S} \frac{\sqrt{\frac{-2\upsilon}{-2\upsilon-\upsilon^2}}}{\sqrt{-2(\upsilon-\upsilon_S)}} d\upsilon \\
&= \sqrt{\frac{R}{g}} \int_{-2}^{\upsilon_S} \frac{1}{\sqrt{-(2+\upsilon)(\upsilon-\upsilon_S)}} d\upsilon \\
&= \sqrt{\frac{R}{g}} \int_{-2}^{\upsilon_S} \frac{1}{\sqrt{2\upsilon_S+(\upsilon_S-2)\upsilon-\upsilon^2}} d\upsilon
\end{aligned}$$

Under the square-root, we can complete the square:

$$\begin{aligned}
T_{min}(\upsilon_S) &= \sqrt{\frac{R}{g}} \int_{-2}^{\upsilon_S} \frac{1}{\sqrt{2\upsilon_S+\frac{(\upsilon_S-2)^2}{4}-\left(\upsilon-\frac{1}{2}(\upsilon_S-2)\right)^2}} d\upsilon \\
&= \sqrt{\frac{R}{g}} \int_{-2}^{\upsilon_S} \frac{1}{\sqrt{\frac{\upsilon_S^2}{4}+\upsilon_S+1-\left(\upsilon-\frac{1}{2}(\upsilon_S-2)\right)^2}} d\upsilon \\
&= \sqrt{\frac{R}{g}} \int_{-2}^{\upsilon_S} \frac{2}{\sqrt{\upsilon_S^2+4\upsilon_S+4-(2\upsilon-(\upsilon_S-2))^2}} d\upsilon
\end{aligned}$$

By another substitution $u = 2\upsilon$, we get:

$$T_{min}(\upsilon_S) = \sqrt{\frac{R}{g}} \int_{-4}^{2\upsilon_S} \frac{1}{\sqrt{(\upsilon_S+2)^2-(u-(\upsilon_S-2))^2}} du$$

Now, we set $\mathcal{U} = u + 2 - \upsilon_S$:

$$T_{min}(\upsilon_S) = \sqrt{\frac{R}{g}} \int_{-2-\upsilon_S}^{2+\upsilon_S} \frac{1}{\sqrt{(\upsilon_S+2)^2-\mathcal{U}^2}} d\mathcal{U}$$

We can rename $s := \upsilon_S + 2$ to get a much nicer looking integral:

$$\begin{aligned} T_{min}(s) =& \sqrt{\frac{R}{g}} \int_{-s}^{s} \frac{1}{\sqrt{s^2 - \mathcal{U}^2}} \, d\mathcal{U} \\ =& \sqrt{\frac{R}{g}} \int_{-s}^{s} \frac{1}{\sqrt{1 - \frac{\mathcal{U}^2}{s^2}}} \, \frac{d\mathcal{U}}{s} \end{aligned}$$

There is one last substitution to do. With $\nu = \frac{\mathcal{U}}{s}$, we finally arrive at

$$\begin{aligned} T_{min} =& \sqrt{\frac{R}{g}} \int_{-1}^{1} \frac{1}{\sqrt{1 - \nu^2}} \, d\nu \\ =& \sqrt{\frac{R}{g}} \int_{1}^{-1} \frac{-1}{\sqrt{1 - \nu^2}} \, d\nu \end{aligned}$$

which is **not dependent of the starting position**. (Since s disappeared from our equation.)

The integral can explicitly be solved, as the integrand is just the derivative of the $\arccos$-function we have seen before.

Using that, we get

$$T_{min} = \sqrt{\frac{R}{g}} \left(\arccos\left(-1\right) - \arccos\left(1\right) \right) = \pi \sqrt{\frac{R}{g}}.$$

So, this is our travel time to the minimum:

$$\boxed{T_{min} = \pi \sqrt{\frac{R}{g}}}$$

Since this is the result for any given starting position, the curve is, indeed, a tautochrone.

q.e.d.