Variational calculus problems part 1 - From Lagrangian to Newtonian mechanics

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Translational motion

Consider a point mass that is moving freely at constant speed v. The actual mass be m.

That makes the kinetic energy

$$E_{kin} = \frac{1}{2} m v^2.$$

Now, let us place that point mass inside a cartesian coordinate frame with arbitrary coordinates x, y, z and define its location vector at any given time t to be

$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

The velocity vector being the location vector's first derivative with respect to time t becomes:

$$\dot{\vec{x}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix}$$

We can generalize the kinetic energy to read:

$$T := E_{kin} = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

Notice, we renamed the kinetic energy as T, as is convention in the Lagrangian formalism to indicate the kinetic energy to be due to translational motion.

Free particle

If there are no forces acting on the mass, there are no other energies to consider.

The **action** functional becomes:

$$S = \int T \, dt$$

From the **least action principle**, we know, that the function(s), the integrand depends on, must minimize the functional, i. e. the integrand must satisfy the Euler-Lagrange-equation(s). Here, these are $x(t)$, $y(t)$ and $z(t)$:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} &= \frac{\partial T}{\partial x} \\
\frac{d}{dt}\frac{\partial T}{\partial \dot{y}} &= \frac{\partial T}{\partial y} \\
\frac{d}{dt}\frac{\partial T}{\partial \dot{z}} &= \frac{\partial T}{\partial z}
\end{aligned}
$$

Since T is symmetric in all the coordinates x, y and z and their respective derivatives, let us just focus on x:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} &= \frac{\partial T}{\partial x} \\
\frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left[\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)\right] &= \frac{\partial}{\partial x}\left[\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)\right] \\
\frac{d}{dt}\left[m\dot{x}\right] &= 0 \\
\ddot{x} &= 0
\end{aligned}
$$

Interesting. We just reproduced two insights from Newtonian mechanics.

The last equation states, that for a free particle, there is no acceleration, while the second-last-equation states, that

$$p_x := m\dot{x},$$

i. e. the **momentum** in x-direction must be constant, i. e. the momentum is a conserved quantity.

Of course, this holds for the other coordinates as well due to the symmetry of the translational kinetic energy T in all these coordinates.

So for a free particle, minimizing $S = \int T \, dt$ reproduces Newtonian physics.

Particle inside a conservative potential

We can step it up by introducing potential energies that we call $V(\vec{x}, \dot{\vec{x}}, t)$ as is, again, convention.

Next, we set the action to be

$$
\begin{aligned}
S &= \int T - V \, dt \\
&=: \int \mathcal{L} \, dt.
\end{aligned}
$$

Here,

$$\mathcal{L} = T - V$$

is the so-called Lagrange function that must satisfy the Euler-Lagrange-equation.

Still using the same kinetic energy T and our "example coordinate" x, we get:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} &= \frac{\partial \mathcal{L}}{\partial x} \\
\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}} - \frac{\partial V}{\partial \dot{x}}\right] &= -\frac{\partial V}{\partial x} \\
\frac{d}{dt}\left[m\dot{x} - \frac{\partial V}{\partial \dot{x}}\right] &= -\frac{\partial V}{\partial x} \\
m\ddot{x} &= -\frac{\partial V}{\partial x} + \frac{d}{dt}\frac{\partial V}{\partial \dot{x}}
\end{aligned}
$$

A very general case of potentials is potentials of **conservative force fields**. One implication of conservative forces is, that the total forces field $\vec{F}$ is just the negative gradient of the potential that shall not depend on any of the velocities.

$$\vec{F} = -\nabla V$$

For our case, this yields

$$\frac{\partial V}{\partial \dot{x}} = 0$$

and

$$-\frac{\partial V}{\partial x} = F_x,$$

where F_x is the x-component of the total force field.

Taking into account the other two coordinates, we get

$$
\begin{aligned}
m\ddot{x} =& F_x \\
m\ddot{y} =& F_y \\
m\ddot{z} =& F_z
\end{aligned}
$$

which can be combined into one equation:

$$\boxed{m\ddot{\vec{x}} = \dot{\vec{p}} = \vec{F}}$$

This is just Newton's second law of motion!

Very nice. Using the least-action-principle for *conservative force fields* with Lagrange function $\mathcal{L} = T - V$ brought us back to Newtonian physics, again.

Consider once again that if any component of the force $\vec{F}$ is zero, the momentum's component into this direction is a constant.

Since the force component is zero if and only if the potential's partial derivative with respect to this coordinate is zero, we know that there is a "momentum-like" quantity for that system that is conserved, if the Lagrange function (or, in most cases, the potential energy) does not depend on one of the coordinates. (I. e. its partial derivative vanishes.)

This is famously known as Noether's theorem.

Non-conservative force fields with non-zero curl

Now, what cases did we not cover?

Well, not every force field can be expressed as (the negative of) a potential's gradient. This does not work anymore, if the force field has non-zero curl. So, a force field is only conservative if

$$\vec{\nabla} \times \vec{F} = \vec{0}.$$

Where $\vec{\nabla}$ is just the nabla operator in three dimensions. (The vector symbol is just a reminder in the notation, that it is, indeed, a vector operator.)

So, since

$$\vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix},$$

$$\vec{\nabla} \times \vec{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} \partial_y F_z - \partial_z F_y \\ \partial_z F_x - \partial_x F_z \\ \partial_x F_y - \partial_y F_x \end{pmatrix}$$

which has to be $\vec{0}$ in a conservative force field, as stated.

> Let us assume we have a force field that is the negative gradient of a scalar potential. Let us calculate the curl:
>
> $$\begin{aligned} \vec{\nabla} \times \vec{F} =& \vec{\nabla} \times \left(-\vec{\nabla} V\right) \\ =& -\vec{\nabla} \times \vec{\nabla} V \\ =& -\vec{\nabla} \times \begin{pmatrix} \partial_x V \\ \partial_y V \\ \partial_z V \end{pmatrix} \\ =& -\begin{pmatrix} \partial_y \partial_z V - \partial_z \partial_y V \\ \partial_z \partial_x V - \partial_x \partial_z V \\ \partial_x \partial_y V - \partial_y \partial_x V \end{pmatrix} \end{aligned}$$
>
> As long as the partial derivative's order can be exchanged, the curl of such a force field will always be $\vec{0}$. There are also some *mathematical theorems* about that.

The elephant in the room: friction

This is the loose end to be discussed.

Recall the expression from earlier

$$m\ddot{x} = -\frac{\partial V}{\partial x} + \frac{d}{dt}\frac{\partial V}{\partial \dot{x}}$$

that was still general, i. e. we did not assume conservative force fields just then.

Let us rewrite that in its more general form that is obtained from the Euler-Lagrange-euquation:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = -\frac{\partial V}{\partial x} + \frac{d}{dt}\frac{\partial V}{\partial \dot{x}}$$

Let us define a new “potential term” D, so that

$$\frac{\partial D}{\partial \dot{x}} = -\frac{d}{dt}\frac{\partial V}{\partial \dot{x}}.$$

That makes our Euler-Lagrange-equation into:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = -\frac{\partial V}{\partial x} - \frac{\partial D}{\partial \dot{x}}$$

(Notice, that we assume the kinetic energy T to still not depend on any of the coordinates.)

$-\frac{\partial V}{\partial x}$ is our regular old conservative force field that in general is the negative gradient of the potential V.

In turn,

$$\frac{\partial D}{\partial \dot{x}} = -\frac{d}{dt}\frac{\partial V}{\partial \dot{x}}$$

must represent present “friction forces” or “dissipative forces” to use a more general term.

Let us consider a mixed potential

$$V = \underbrace{V_C}_{\text{conservative}} + \underbrace{V_D}_{\text{dissipative}}$$

Where V_C still only depends on the coordinates while V_D shall only depend on velocities, i. e.:

$$V = V_C(x) + V_D(\dot{x})$$

That makes our dissipation (friction) force into the following:

$$\begin{aligned} F_D :=& -\frac{\partial D}{\partial \dot{x}} \\ =& +\frac{d}{dt}\frac{\partial V}{\partial \dot{x}} \\ =& \frac{d}{dt}\frac{\partial V_D}{\partial \dot{x}} \end{aligned}$$

For a simple friction force that is proportional to a power of the velocity like

$$F_D = \alpha \dot{x}^n,$$

we find a possible D to generate this friction forces to be:

$$\begin{aligned} D &= -\int F_D \, d\dot{x} \\ &= -\frac{\alpha}{n+1} \cdot \dot{x}^{n+1} + D_0 \end{aligned}$$

D_0 is just an arbitrary constant from the integration. Since in using the Euler-Lagrange-equation, we will always have to take the derivative of D again, we can just drop D_0 and construct a friction force that is proportional to $\dot{x}^n$ with a “generator” D_n like:

$$\boxed{D_n = -\frac{\alpha}{n+1} \cdot \dot{x}^{n+1}}$$

> *Now, can we also find the proper “friction potential” V_D?*
>
> By definitions,
>
> $$F_D = \frac{d}{dt}\left(\frac{\partial V_D}{\partial \dot{x}}\right).$$
>
> In turn, we can rewrite
>
> $$V_D = \int \int F_D \, dt \, d\dot{x}$$
>
> “undoing” the derivatives.
>
> **Be aware, that we cannot exchange the order of integrations, as we cannot exchange the original partial and total derivatives.** (Partial with respect to $\dot{x}$, total with respect to t.)
>
> This is a mayor complication as we cannot give a general solution for this double integral

$$\begin{aligned} V_D =& \int \int F_D \, dt \, d\dot{x} \\ =& \int \int \alpha \dot{x}^n \, dt \, d\dot{x} \\ =& \alpha \int \int \dot{x}^n \, dt \;\; d\dot{x}. \end{aligned}$$

We can do further simplifications for $n = 1$, which can be considered a “demo case”:

$$\begin{aligned} V_D(n = 1) =& \alpha \int \int \dot{x} \, dt \, d\dot{x} \\ =& \alpha \int x + c_1 \, d\dot{x} \\ =& \alpha x \dot{x} + c_1 \dot{x} + c_2(x, t) \end{aligned}$$

This is inconsistent with our original construction that assumed V_D to be not depending on the actual coordinate x, as we would get a contribution to the “conservative part” since $\frac{\partial V_D}{\partial x} \neq 0$.

So, if there is any hope of finding a solution, it will be way more complicated.

Furthermore, notice, that c_1 must be an actual constant as it arises from the time integration that “undid” a total derivative, while c_2 can actually be depending on x and t, as they are considered constants with respect to the “partial” $\dot{x}$-integration. (Since taking the partial derivative later will kill that term again.)

Now, can we use that detail to construct a consistent dissipative potential term that does not contribute to the “conservative” forces?

To achieve this, we would need

$$\begin{aligned} 0 =& \frac{\partial V_D}{\partial x} \\ =& \alpha \dot{x} + \frac{\partial c_2(x, t)}{\partial x} \\ \Rightarrow \quad c_2(x) =& - \int \alpha \dot{x} \, dx \end{aligned}$$

So can we just use

$$c_2 = -\alpha \dot{x} x?$$

No, this does not work, because when applying the derivatives to a so-constructed V_D to calculate our force terms, we just get 0, because

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}(c_1 \dot{x}) = \frac{d}{dt}c_1 = 0.$$

Apparently, we need to consider $\dot{x}$ to be a function of x itself for this integration, which is somewhat reasonable, as a given $x(t)$ forces our hand into deriving a particular $\dot{x}(t)$, which is notably a function of time, as is x itself.

In turn, it stands to reason that at least for invertible $x(t) \Rightarrow t(x)$, we can consider

$$\dot{x} = \dot{x}(t) = \dot{x}(t(x)) = \dot{x}(x).$$

(Which gets even messier in higher dimensions, by the way.)

Alright. It might not be outright impossible to find a potential V_D that generates appropriate dissipative forces for a given problem, but constructing it from familiar friction forces for usage in the Lagrange formalism is not feasible as it would require us to already know a relation between $\dot{x}$ and x, which is most likely what we want to actually obtain from the Lagrange function.

If we had a relation $\dot{x}(x) =: g(x)$, we could find our function of interest $x(t)$ much easier:

$$\begin{aligned} \dot{x} = \frac{dx}{dt} &= g(x) \\ \Rightarrow \quad \frac{1}{g(x)}dx &= dt \\ \Rightarrow \quad t(x) = \int dt &= \int \frac{1}{g(x)}dx \end{aligned}$$

i. e. we would already be at a point where we no longer need to do much to find the relation between x and t, except solving an integral and inverting the function

$$t(x) = \int \frac{1}{g(x)}dx$$

that arises from this integral, and in turn, we do not need to do any of the Lagrange stuff, anymore.

> So, this is not helpful in practice and if we really want to use the Lagrange formalism for a problem where friction is a concern, we best go with the D_n-generator-term and adjust the Euler-Lagrange-equation accordingly.

In conclusion, we get a modified Euler-Lagrange-equation with the aforementioned "friction generator" D_n. (Assuming the rest of the potential energies to be conservative or at the very least, not interfering with the friction generator.)

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x} - \frac{\partial D_n}{\partial \dot{x}}}$$

We will not discuss different forms of friction or dissipation here, as this is rarely solved with the Lagrangian formalism when dealing with real-world problems.

Summary

While we might have gotten lost in some of the details, we achieved quite a lot.

We demonstrated that we can define the action S as integral over the Lagrange function $\mathcal{L}$ like

$$S = \int \mathcal{L}\, dt,$$

where the Lagrange function is defined as the difference between translational kinetic energy T and the scalar potential V that gives rise to all the conservative forces:

$$\mathcal{L} = T - V$$

With this construction, minimizing the action functional using the Euler-Lagrange-equation(s) reproduces Newton's laws of motion and even conservation of momentum via Noether's theorem.

If q is any of the coordinates used to describe the system, the Euler-Lagrange equation for that coordinate reads:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$$

In the case of $\mathcal{L}$ being not explicitly depending on q, the **canonic momentum** $p_q := \frac{\partial \mathcal{L}}{\partial \dot{q}}$ is a conserved quantity per Noether's theorem:

$$\frac{\partial \mathcal{L}}{\partial q} = 0 \;\Rightarrow\; p_q = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \text{const.}$$

Also, we have seen limitations to the Lagrange framework like force fields with non-zero curl or friction.

However, at least for the one-dimensional case, we managed to extend the Euler-Lagrange-equation to reproduce the proper friction forces, adding a “generator term”:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial D}{\partial \dot{q}}$$

In the end, it is at least conceivable that actions might exist that properly reproduce Newtonian physics even in these cases where our approaches broke down so far.

Indeed, the least-action-principle is considered to be a fundamental principle in all disciplines of physics, as people have managed to find workable actions not only for classical mechanics but also quantum mechanics and relativity.

Since most of the time, actions are not obtained from physical insight but instead, constructed to reproduce the “desired” physical results, one might argue that this might not be something that is actually handed to us by nature but instead, it is the basis for the powerful Lagrange formalism that can, if properly chosen, give us reasonable physics as results of our calculations.

Personally, I believe that at some point, with a better understanding of physics, mankind will have a better idea of how the action terms are linked to our reality, but I will gladly admit that this is more philosophy than physics.