Parabolic mirror

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Jan Bergner

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Parabolic mirrors in the wild

Parabolic mirrors have a lot of technical applications. They are used as sattelite dishes, solar cookers or headlamps which is due to one unique property.

On wikipedia, you can even find an application as lighter for the Olympic Flame.

Figure 1: Lighting the Olympic Flame

Parabolic mirrors reflect parallel light coming from one direction in such a way, that all reflected beams hit one point, which is called focal point or *focus*.

In fact, this is one of the possible defining properties of a parabolic mirror, which will, as the name suggests, have the shape of a paraboloid.

Or will it?

I recall being told at school that the catenary was a parabola while in fact, it is a hyperbolic cosine, which can just be approximated as a parabola. But it is not the same!

So I guess, I'll have to check whether I have been tricked or not...

Geometry

Of course, we start with a sketch:

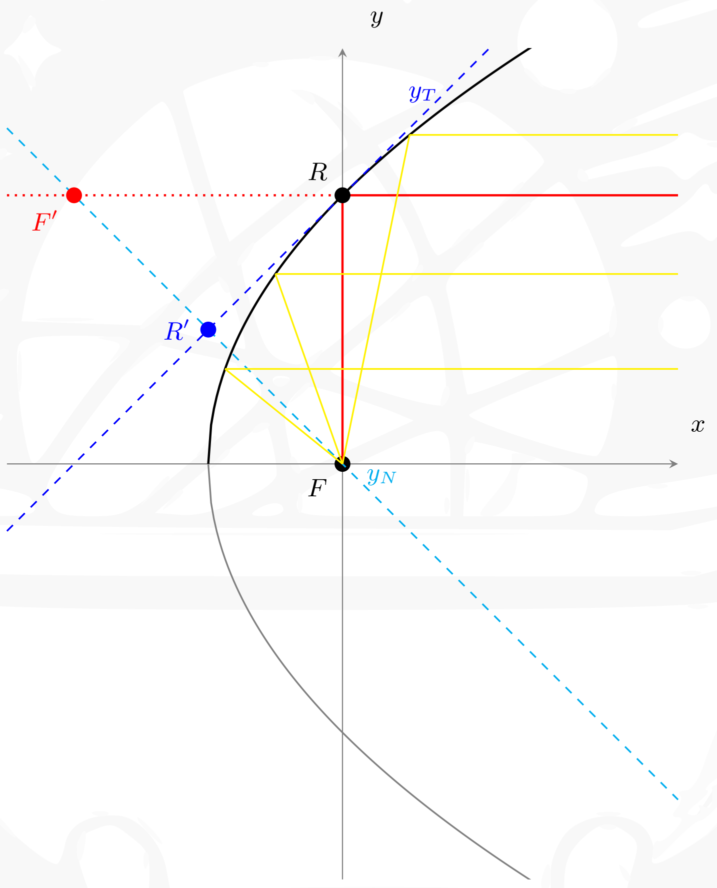

Figure 2: Geometry of a parabolic mirror

Light (yellow) is coming from the right, i. e. it goes in negative x-direction.

The mirror is represented by the black curve above the x-axis. The gray curve below the x-axis is actually also representing the mirror, but we will focus on the upper part, since both parts will be symmetric, anyway. (In fact, it is sufficient to let the black upper part rotate around the x-axis to create the full mirror.)

The focus point F must rest on the x-Axis, because of the symmetry of the setup.

We will consider one particular ray of light. (red)

It will hit the mirror and be reflected towards the focus. The reflection point is R.

Now, how does a reflection at a curve work?

It will be the same as reflecting over the curve's tangent at R. This tangent y_T is shown in the sketch as a dashed blue line.

However, we can also imagine the focus F to be reflected over the same tangent. The reflection of F would not go through R, but through another point R' on the tangent and land on the "virtual" focus point F'. As the dashed red "virtual" light ray indicates, we can imagine the original ray to go straight through the mirror and follow a straight line to hit the virtual focus F'.

Now, there is some easy geometry nailing down the location of R'. It will be at the (orthogonal) intersection of the tangent y_T and the straight line y_N connecting focus and virtual focus.

Indeed, y_N is perpendicular to y_T and the distances between R' and F or F', respectively, will be the same. Furthermore, this means that the difference of their coordinates is also identical, i. e.:

$$
\begin{aligned}
x_{R'} - x_{F'} &= x_F - x_{R'} \\
y_{F'} - y_{R'} &= y_{R'} - y_F
\end{aligned}
$$

Summing up

1. Let a (red) ray of light go into the mirror. It will hit the reflecting surface at point R.
2. It shall be reflected into the focus F.
3. Construct the mirror's tangent y_T through R and reflect the focus over that tangent onto a "virtual" focus F'.
4. In order to actually hit the real focus F, the straight-line-continuation of the incoming light ray must hit the "virtual" focus F'.
5. When reflecting the focus over the tangent, the actual reflection will happen at a point R', which will be on the tangent.
6. Furthermore, R' will have the same distance to the focus and the "virtual" focus and the straight line y_N connecting all three of them will be perpendicular to the tangent y_T.

The geometry in equations

Now, let us put some numbers on that.

A straight line always has an equation of the form

$$\begin{aligned} y(x) =& mx + b \\ =& m(x - x_0) + y_0, \end{aligned}$$

where m is the slope and (x_0, y_0) is any point on that straight line.

Therefore, we can express the tangent $y_T(x_T)$ and the perpendicular straight line $y_N(x_N)$ as:

$$\begin{aligned} y_T - y_R =& m_T(x_T - x_R) \\ y_N - y_F =& m_N(x_N - x_F) \end{aligned}$$

For the sake of simplicity, we will place the focus into the origin, i. e.:

$$(x_F, y_F) = (0, 0)$$

This will come at the cost of x ranging into the negative numbers and we will get back to that.

Furthermore, the slopes of perpendicular lines always have a product of -1:

$$\begin{aligned} m_T \cdot m_N =& -1 \\ \Leftrightarrow \;\; m := m_T =& -\frac{1}{m_N} \\ \Leftrightarrow \;\; m_T = m, \;\; m_N =& -\frac{1}{m} \end{aligned}$$

Therefore,

$$y_N = -\frac{1}{m}x_N \;\; \Rightarrow \;\; x_N = -my_N$$

and

$$y_T = m(x_T - x_R) + y_R.$$

The point R' is the intersection of y_T and y_N. Therefore, it will satisfy both equations:

$$
\begin{aligned}
x_{R'} =& -my_{R'}\\
y_{R'} =& m(x_{R'} - x_R) + y_R
\end{aligned}
$$

Recall

$$
\begin{aligned}
y_{F'} - y_{R'} =& y_{R'} - \underbrace{y_F}_{=0}\\
y_{F'} =& 2y_{R'}\\
y_{R'} =& \frac{y_{F'}}{2}.
\end{aligned}
$$

Additionally, we know that F' is on the straight-line-continuation of the original light ray going into negative x-direction, i. e. All points on the light ray have the same y-coordinate. Since the actual reflection point R is hit by that beam of light,

$$y_{F'} = y_R.$$

So finally,

$$y_{R'} = \frac{y_R}{2}.$$

Combining all together,

$$
\begin{aligned}
y_{R'} =& \frac{y_R}{2}\\
x_{R'} =& -my_{R'}\\
y_{R'} =& m(x_{R'} - x_R) + y_R.
\end{aligned}
$$

We can insert the first two equations into the last one:

$$
\begin{aligned}
y_{R'} =& m(x_{R'} - x_R) + y_R \\
\frac{y_R}{2} =& m\left[(-m y_{R'}) - x_R\right] + y_R \\
\frac{y_R}{2} =& m\left[\left(-m\left(\frac{y_R}{2}\right)\right) - x_R\right] + y_R \\
\frac{y_R}{2} - y_R =& \left(-m^2\left(\frac{y_R}{2}\right)\right) - m x_R \\
-\frac{1}{2} y_R =& -\frac{m^2}{2} y_R - m x_R \\
(m^2 - 1) y_R =& -2 m x_R
\end{aligned}
$$

At this point, we are tantalizingly close to our original problem. (To re-iterate, we want to know the exact shape of the parabolic mirror, i. e. the curve $y(x)$, that will reflect all parallel rays of light going into negative-x-direction into one focus point F.)

Since R is part of the mirror and so far we selected it arbitrarily, we can replace

$$
x_R \to x, \quad y_R \to y.
$$

Now, what about m?

Remember, that m was the slope of the tangent y_T through R, which is the same as the first derivative $y'(x)$ of the function $y(x)$ we are looking for. (In fact, this is the definition of the first derivative.)

So, we can also replace

$$
m \to y'
$$

to get:

$$
\begin{aligned}
(m^2 - 1) y_R =& -2 m x_R \\
\left(y'^2 - 1\right) y =& -2 y' x \\
y'^2 + 2 y' \frac{x}{y} - 1 =& 0
\end{aligned}
$$

Notice that in the last step, we divided by y, which means we cannot have $y = 0$. Strictly speaking, we cannot be sure that whatever we find down the road will satisfy the case $y = 0$.

But that is not a problem, as the incoming ray at $y = 0$ will hit the focus even before hitting the mirror. So, we can exclude this case for now.

Thus, we are left with a quadratic equation in y':

$$\begin{aligned} {y'}^2 + 2y'\frac{x}{y} - 1 =& 0 \\ y' =& -\frac{x}{y} \pm \sqrt{1 + \left(\frac{x}{y}\right)^2} \end{aligned}$$

Now, we have to answer the age-old question arising from quadratic equations. Which solution will we go with? The one with the plus-sign in front of the square-root or the one with the minus-sign?

Well, we know that y will be positive, since we constructed it that way. The same goes for the slope y'.

Mathematically, it is not absolutely clear that the slope can never be negative, but physically it is reasonable to assume this.

The square $\left(\frac{x}{y}\right)^2$ is always positive, since x and y are real values. Since below the square-root, we add a positive 1 to the square, the whole square-root will be positive and larger than the absolute value of the fraction:

$$\sqrt{1 + \left(\frac{x}{y}\right)^2} > \left|\frac{x}{y}\right|$$

When $x > 0$, the $-\frac{x}{y}$-term is negative and since we need an overall positive slope, the square-root-term must be positive.

When $x < 0$, the $-\frac{x}{y}$-term is positive, but the square-root-term still cannot be negative, since its absolute value is larger and would turn the whole term into a net negative.

So, we can comfortably choose the plus-sign.

$$\boxed{y' = -\frac{x}{y} + \sqrt{1 + \left(\frac{x}{y}\right)^2}}$$

Yaay, we got an ordinary non-linear first-order differential equation!

Our problem is basically solved!

Solving the differential equation

Indeed, this differential equation is not as evil as it looks. There is one sneaky substitution, we will have to do, but other than that, it is not really difficult.

In order to solve

$$y' = -\frac{x}{y} + \sqrt{1 + \left(\frac{x}{y}\right)^2},$$

the first step is introducing the substitution

$$u := \frac{x}{y} \;\Rightarrow\; y = \frac{x}{u}.$$

Notice, that for $u = 0, x = 0$ and thus, the fraction $\frac{x}{u}$ will still have a well-defined limit of y from both sides. That will allow us to ignore the issue of a zero in the denominator in this case.

Next, we calculate the derivative

$$y' = \frac{u - xu'}{u^2}$$

and insert into the original differential equation:

$$\begin{aligned} \frac{u - xu'}{u^2} &= -u + \sqrt{1 + u^2} \\ u - xu' &= -u^3 + u^2\sqrt{1 + u^2} \\ u + u^3 - u^2\sqrt{1 + u^2} &= xu' \\ \frac{1}{x} &= \frac{u'}{u + u^3 - u^2\sqrt{1 + u^2}} \end{aligned}$$

Now, we have a completely separable differential equation. We split the derivative like

$$u' = \frac{du}{dx}$$

and integrate on both sides:

$$\int \frac{dx}{x} = \underbrace{\int \frac{du}{u + u^3 - u^2\sqrt{1 + u^2}}}_{=:I}$$

The left-hand-side is easily solved by

$$\int \frac{dx}{x} = \ln|x| + c,$$

while the right-hand-side is a little more involved. That is why we call the whole experssion I and do an auxiliary calculation. Afterwards, we will re-insert into

$$\ln|x| + c = I.$$

Auxiliary calculation of the integral I

We need to solve

$$I = \int \frac{du}{u + u^3 - u^2\sqrt{1+u^2}}$$

To do that, we want to get rid of the square-root. This can be achieved by introducing Euler's first substitution:

$$u =: \frac{s^2-1}{2s}$$

Yet again, we must fear a zero in the denominator. But in this case, we are in luck.

Let us calculate s in terms of u:

$$\begin{aligned} u =& \frac{s^2-1}{2s} \\ 2su =& s^2 - 1 \\ 0 =& s^2 - 2su - 1 \\ \Rightarrow \quad s =& u \pm \sqrt{u^2+1} \end{aligned}$$

Now let us assume, $s = 0$:

$$\begin{aligned} 0 =& u \pm \sqrt{u^2+1} \\ -u =& \pm \sqrt{u^2+1} \quad | (\cdots)^2 \\ u^2 =& u^2+1 \\ 0 =& 1 \perp \end{aligned}$$

This is a contradiction and so, s cannot be zero. Our substitution can therefore progress and we, once again, calculate the first derivative:

$$\begin{aligned} u =& \frac{s^2-1}{2s} \\ \Rightarrow \quad u' =& \frac{(2ss')(2s)-(s^2-1)(2s')}{4s^2} \\ =& s' \frac{2s^2-s^2+1}{2s^2} \\ =& s' \frac{s^2+1}{2s^2} \end{aligned}$$

Now, what will happen to the square-root $\sqrt{1+u^2}$?

We introduce the substitution for the expression inside the square-root:

$$\begin{aligned} 1+u^2 =& 1+\left(\frac{s^2-1}{2s}\right)^2 \\ =& \frac{4s^2}{4s^2}+\frac{\left(s^2-1\right)^2}{4s^2} \\ =& \frac{4s^2}{4s^2}+\frac{s^4-2s^2+1}{4s^2} \\ =& \frac{s^4+2s^2+1}{4s^2} \\ =& \left(\frac{s^2+1}{2s}\right)^2 \end{aligned}$$

Hooray! Thanks to the great **Leonhard Euler**, we managed to forge the term inside the square-root into a perfect square, which will allow us to resolve the square-root.

Recall

$$I = \int \frac{du}{u + u^3 - \underbrace{u^2\sqrt{1+u^2}}_{=:R}}$$

and let us calculate the summands in terms of s, beginning with the evil square-root-term:

$$\begin{aligned} R =& u^2\sqrt{1+u^2} \\ =& \left[(1+u^2) - 1\right] \cdot \sqrt{1+u^2} \\ =& \left[\left(\frac{s^2+1}{2s}\right)^2 - 1\right] \cdot \frac{s^2+1}{2s} \\ =& \frac{s^4 - 2s^2 + 1}{4s^2} \cdot \frac{s^2+1}{2s} \\ =& \frac{s^6 - s^4 - s^2 + 1}{8s^3} \end{aligned}$$

Since

$$u = \frac{s^2-1}{2s},$$

$$\begin{aligned} u^3 =& \left(\frac{s^2-1}{2s}\right)^3 \\ =& \frac{s^6 - 3s^4 + 3s^2 - 1}{8s^3}. \end{aligned}$$

We are getting somewhere!

Let us simplify the whole denominator

$$D := u + u^3 - R$$

like so:

$$
\begin{aligned}
D =& u + u^3 - R \\
=& u + u^3 - u^2\sqrt{1+u^2} \\
=& \left(\frac{s^2-1}{2s}\right) + \left(\frac{s^6-3s^4+3s^2-1}{8s^3}\right) - \left(\frac{s^6-s^4-s^2+1}{8s^3}\right) \\
=& (8s^3)^{-1}\left[4s^2(s^2-1) + (s^6-3s^4+3s^2-1) - (s^6-s^4-s^2+1)\right] \\
=& (8s^3)^{-1}\left[4s^4-4s^2+s^6-3s^4+3s^2-1-s^6+s^4+s^2-1\right] \\
=& (8s^3)^{-1}\left[2s^4-2\right] \\
=& \frac{s^4-1}{4s^3}
\end{aligned}
$$

At this point, we can smell success, but we must not forget the numerator of the integral I's integrand, i. e. the du-term, we still have to transform.

But we already did this with

$$
u' = s'\frac{s^2+1}{2s^2}.
$$

Now, we can express the full integrand in terms of s:

$$
\begin{aligned}
\frac{du}{D} =& \frac{ds\frac{s^2+1}{2s^2}}{\frac{s^4-1}{4s^3}} \\
=& \frac{(s^2+1)\cdot 4s^3}{2s^2\cdot(s^4-1)}ds \\
=& \frac{2s}{s^2-1}ds
\end{aligned}
$$

Uuuhhh, this is beautiful! Let us cherish this moment!

$$
\begin{aligned}
I &= \int \frac{2s}{s^2-1}ds \\
&= \ln\left|s^2-1\right|
\end{aligned}
$$

Undoing substitutions

As we had

$$\ln|x| + c = I,$$

this implies:

$$\begin{aligned} \ln|x| + c &= \ln\left|s^2 - 1\right| \\ \Rightarrow \; e^{\ln|x|} \cdot \underbrace{e^c}_{\alpha} &= e^{\ln\left|s^2-1\right|} \\ \alpha \cdot |x| &= \left|s^2 - 1\right|. \end{aligned}$$

Notice that α will be positive, as it is e to the something, which for a real something will always be positive.

Now, it would be so tempting, to just ignore the absolute values and it might even lead to the correct solution, but we will be thorough.

Recall Euler's substitution, and its implications:

$$\begin{aligned} u &= \frac{s^2 - 1}{2s} \\ s^2 - 1 &= 2su \\ s &= u \pm \sqrt{1 + u^2} \end{aligned}$$

We will now insert.

$$\begin{aligned} \alpha\,|x| &= \left|s^2 - 1\right| \\ \alpha\,|x| &= |2su| \end{aligned}$$

To get rid of the absolute values, we square both sides of the equation.

$$\begin{aligned}(\alpha\,|x|)^2 &= (|2su|)^2\\ \alpha^2x^2 &= 4s^2u^2\\ \alpha^2x^2 &= 4(s^2-1)u^2+4u^2\\ \alpha^2x^2 &= 4u^2\cdot 2su+4u^2\\ &= 4u^2\cdot 2u\left(u\pm\sqrt{1+u^2}\right)+4u^2\\ &= 8u^4\pm 8u^3\sqrt{1+u^2}+4u^2\\ \left(\frac{\alpha}{2}\right)^2x^2 &= 2u^4\pm 2u^3\sqrt{1+u^2}+u^2\end{aligned}$$

Since up to this point, α is just an undefined integration constant without a lot of meaning, we will replace

$$\frac{\alpha}{2}=\eta$$

to keep the equations cleaner. Additionally, we can re-insert

$$u=\frac{x}{y}.$$

$$\begin{aligned}\left(\frac{\alpha}{2}\right)^2x^2 &= 2u^4\pm 2u^3\sqrt{1+u^2}+u^2\\ \eta^2x^2 &= 2u^4\pm 2u^3\sqrt{1+u^2}+u^2\\ &= 2\left(\frac{x}{y}\right)^4\pm 2\left(\frac{x}{y}\right)^3\sqrt{1+\left(\frac{x}{y}\right)^2}+\left(\frac{x}{y}\right)^2\end{aligned}$$

The next step is multiplying by y^4 and canceling x^2, as it is a factor that appears in every summand. Once again, we will have to be careful about the case of $x=0$.

$$\begin{aligned}\eta^2x^2 &= 2\left(\frac{x}{y}\right)^4\pm 2\left(\frac{x}{y}\right)^3\sqrt{1+\left(\frac{x}{y}\right)^2}+\left(\frac{x}{y}\right)^2\\ \eta^2x^2y^4 &= 2x^4\pm 2x^3\sqrt{y^2+x^2}+x^2y^2\\ \eta^2y^4 &= 2x^2\pm 2x\sqrt{y^2+x^2}+y^2\end{aligned}$$

Next, we want to get rid of the $\pm$-sign. The way to do this is isolating that term and squaring it once again:

$$\begin{aligned} \eta^2 y^4 - 2x^2 - y^2 &= \pm\, 2x\sqrt{y^2 + x^2} \quad |(\cdots)^2 \\ \left[\eta^2 y^4 - 2x^2 - y^2\right]^2 &= 4x^2\left(y^2 + x^2\right) \\ \eta^4 y^8 - 4\eta^2 x^2 y^4 - 2\eta^2 y^6 + 4x^4 + 4x^2 y^2 + y^4 &= 4x^2 y^2 + 4x^4 \end{aligned}$$

Yet again, we got lucky. The whole right-hand-side cancels against corresponding terms on the left-hand-side and afterwards, we can draw a factor of y^4 to the front. Since we already excluded $y = 0$, we can cancel this one as well:

$$\begin{aligned} \eta^4 y^8 - 4\eta^2 x^2 y^4 - 2\eta^2 y^6 + 4x^4 + 4x^2 y^2 + y^4 &= 4x^2 y^2 + 4x^4 \\ \eta^4 y^8 - 4\eta^2 x^2 y^4 - 2\eta^2 y^6 + y^4 &= 0 \\ y^4\left[\eta^4 y^4 - 4\eta^2 x^2 - 2\eta^2 y^2 + 1\right] &= 0 \\ \eta^4 y^4 - 4\eta^2 x^2 - 2\eta^2 y^2 + 1 &= 0 \end{aligned}$$

Now, let us solve for x:

$$\begin{aligned} \eta^4 y^4 - 4\eta^2 x^2 - 2\eta^2 y^2 + 1 &= 0 \\ \eta^4 y^4 - 2\eta^2 y^2 + 1 &= 4\eta^2 x^2 \\ x^2 &= \frac{\left(\eta^2 y^2 - 1\right)^2}{4\eta^2} \\ x^2 &= \left(\frac{\eta}{2} y^2 - \frac{1}{2\eta}\right)^2 \end{aligned}$$

We do one last substitution

$$a = \frac{\eta}{2}$$

and take the square-root:

$$x^2 = \left(\frac{\eta}{2}y^2 - \frac{1}{2\eta}\right)^2$$
$$x^2 = \left(ay^2 - \frac{1}{4a}\right)^2$$
$$\Rightarrow \; x = \pm\left(ay^2 - \frac{1}{4a}\right)$$

Now, what can we do with the last $\pm$-sign to bother us on this one?

Well, if we exchanged the variables x and y, we would face a simple equation for a parabola. That is a good thing, already.

The $\pm$-sign would just tell us, whether its opening is facing upwards or downwards, which is fine for our situation as well, since we only *said*, that the light should come in from the right, i.e. positive-x-direction, but we did not really enforce this in our equations. Thus, the plus-sign corresponds to a parabola "laying on the side" that opens to the right, while the minus-sign would flip it over the y-axis to make it open to the left.

Sort of a "bonus" is, that our solution also covers the parabola-branch below the x-axis, but this is more or less just a detail.

Our problem is generally solved by any function

$$x = ay^2 - \frac{1}{4a},$$

or, if we want to rewrite ond only focus on the upper branch:

$$\boxed{y = \sqrt{\frac{x}{a} + \frac{1}{4a^2}}}$$

After all, they did **not** lie to me at school - this time!

Consistency checks for the result

A first consistency check to judge, whether our result is reasonable is to plot it for some values of a:

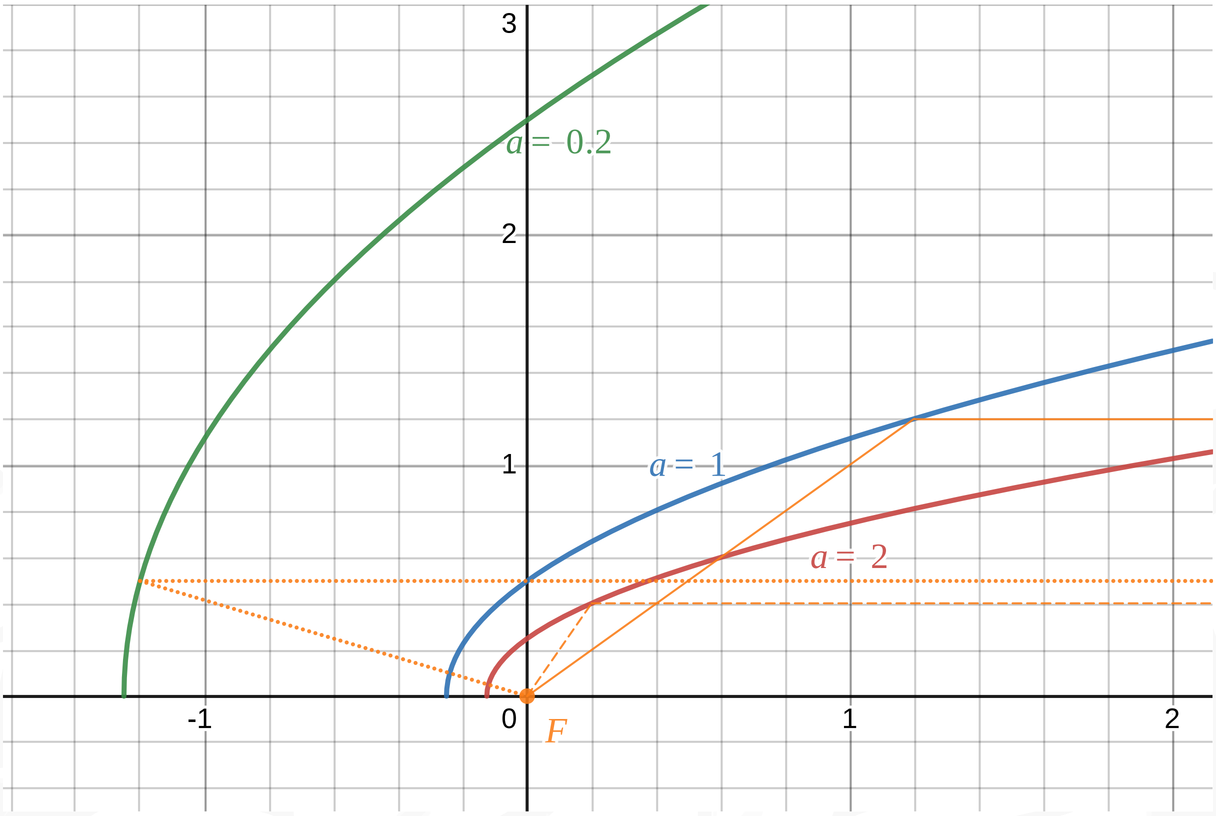

Figure 3: Parabolic mirrors for some parameters.

First of all, the curves are continuous, smooth and monotonic. That is certainly a good start for something that should represent a physical shape.

Furthermore, the “critical” cases $x = 0$ and $y = 0$ do not seem to be singular in our solution.

Critical cases $x = 0$ and $y = 0$

The case $x = 0$ is in no way special in our result.

The curves are still smooth and continuous.

Furthermore, we can have a look at the first derivative(s):

$$\begin{aligned} y'(x) =& \frac{d}{dx} y(x) \\ =& \frac{d}{dx}\left[\frac{1}{\sqrt{a}}\sqrt{x+\frac{1}{4a}}\right] \\ =& \frac{1}{2\sqrt{a}}\frac{1}{\sqrt{x+\frac{1}{4a}}} \end{aligned}$$

This derivative is still smooth and continuous at $x = 0$. Therefore, this case does not seem to present a problem.

Now, what about $y = 0$?

Well, the first derivative goes off to infinity, but that is okay, since this only means that the tangent at this point is vertical.

Furthermore, if we were to use the lower branch, i. e. the function $-y(x)$, the derivative's limit for $y = 0$, i. e. $x = -\frac{1}{4a}$ of the derivative also goes to a vertical tangent. (Notice, that in this case, it would be negative infinity.)

Aside from the formal points, we can argue that the reflection of a light ray hitting the mirror at $y = 0$ will also go through the focus. So, our solution satisfies our requirements, as well for this case.

Thus, our solution, indeed, works all over the place.