An analytical solution of the Navier-Stokes equations for a highly simplified flow problem

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This is a guest article by Christoph Gordalla.

Motivation

The Navier-Stokes equations are often mentioned as impossible to solve analytically. This holds in nearly all practical cases. Nevertheless, using them to analytically solve a highly simplified problem will help students to understand the undergoing processes in fluid dynamics. Furthermore, the example below pronounces a very important skill in solving physical problems: Often the most important thing is to find and exploit symmetries. This will be the most important step in finding a solution of the following problem. For these reasons, it is also highly suited as a task for undergraduate students.

Actual example

The Navier-Stokes equations has the form

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \eta\nabla^2\mathbf{u}.$$

If we furthermore assume the fluid to be incompressible, we also have to take the continuity equation into consideration:

$$\nabla\cdot\mathbf{u} = 0$$

These two equation are ***not*** two single equations but a whole system of PDEs:

$$\begin{aligned}
\rho\left(\partial_t u_x + (u_x\partial_x + u_y\partial_y + u_z\partial_z)u_x\right) &= -\partial_x p + \eta(\partial_x^2 + \partial_y^2 + \partial_z^2)u_x \\
\rho\left(\partial_t u_y + (u_x\partial_x + u_y\partial_y + u_z\partial_z)u_y\right) &= -\partial_y p + \eta(\partial_x^2 + \partial_y^2 + \partial_z^2)u_y \\
\rho\left(\partial_t u_z + (u_x\partial_x + u_y\partial_y + u_z\partial_z)u_z\right) &= -\partial_z p + \eta(\partial_x^2 + \partial_y^2 + \partial_z^2)u_z \\
\partial_x u_x + \partial_y u_y + \partial_z u_z &= 0
\end{aligned}$$

For a general problem, it is nearly impossible to solve this monster! But for highly simplified cases, it is actually possible to solve it analytically. One of those cases is the flow that arises through two plates for a given pressure gradient that drives this flow. Our aim is now to calculate the steady state flow between both plates and especially to get the velocity profile of the flow. Without loss of generality, let us assume that the pressure gradient points into x-direction, thus the flow will also do this. The pressure gradient is moreover constant and can therefore be written as

$$\partial_x p = \frac{\Delta p}{l},$$

where Δp is the pressure loss over the length l. Furthermore, the plates are infinite in x- and z-direction. This will give us a lot of symmetries to exploit! We now have $u_y = u_z = 0$ since ∇p only shows in x-direction. Since the plates are infinite in x- and z direction, no variations in velocity are expected in these directions:

$$\partial_x u_i = \partial_z u_i = 0$$

with $i \in \{x, y, z\}$. And, last but not least, we want to calculate the steady state flow, which results in

$$\partial_t u_i = 0.$$

With these symmetries exploited, our equations will simplify a lot:

$$0 = -\frac{\Delta p}{l} + \eta \partial_y^2 u_x$$

Using the no-slip condition for boundaries results in the boundary conditions:

$$u(D/2) = u(-D/2) = 0,$$

chosing the middle between the two plates as $y = 0$ and naming the plates distance D.

Now we can integrate this equation:

$$\begin{aligned}
\partial_y^2 u_x &= \frac{1}{\eta}\frac{\Delta p}{l} \\
\partial_y u_x &= \frac{1}{\eta}\frac{\Delta p}{l} y + C_1 \\
u_x(y) &= \frac{1}{2\eta}\frac{\Delta p}{l} y^2 + C_1 y + C_2
\end{aligned}$$

Using the boundary condition gives two equations to determine C_1 and C_2:

$$u_x\left(\frac{D}{2}\right) = \frac{1}{2\eta}\frac{\Delta p}{l}\left(\frac{D}{2}\right)^2 + C_1\frac{D}{2} + C_2 = 0$$
$$u_x\left(\frac{D}{2}\right) = \frac{1}{2\eta}\frac{\Delta p}{l}\left(\frac{D}{2}\right)^2 - C_1\frac{D}{2} + C_2 = 0$$

Subtracting both equations gives us $C_1 = 0$. Adding both equations leads to:

$$0 = \frac{1}{\eta}\frac{\Delta p}{l}\left(\frac{D}{2}\right)^2 + 2C_2$$
$$\Rightarrow C_2 = \frac{1}{2\eta}\frac{\Delta p}{l}\left(\frac{D}{2}\right)^2$$

Inserting this into the solution leads to:

$$u_x(y) = \frac{1}{2\eta}\frac{\Delta p}{l}\left(y^2 - \left(\frac{D}{2}\right)^2\right),$$

which is the well known parabolic velocity profile between two plates.

Note that $u_x(D)$ is negative. This is in accordance to reality since the fluid flows from high pressure values to low pressure values (thus in the direction of $-\Delta p/l$). To pronounce this fact, one can rewrite the equation above to:

$$u_x(y) = \frac{1}{2\eta}\left(-\frac{\Delta p}{l}\right)\left(\left(\frac{D}{2}\right)^2 - y^2\right)$$

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