Looking out of the crow’s nest

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Jan Bergner

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In ancient times, sailing ships had a high mast with a so-called "**crow's nest**", where a member of the crew could climb up and had a good view over the ocean.

This raises a question.

How far can you see from the crow's nest?

Assuming you know your altitude above the sea inside the crow's nest, it is possible to calculate the distance to the horizon. (Assuming Earth to be a perfect sphere, that is.)

Let us begin with a sketch.

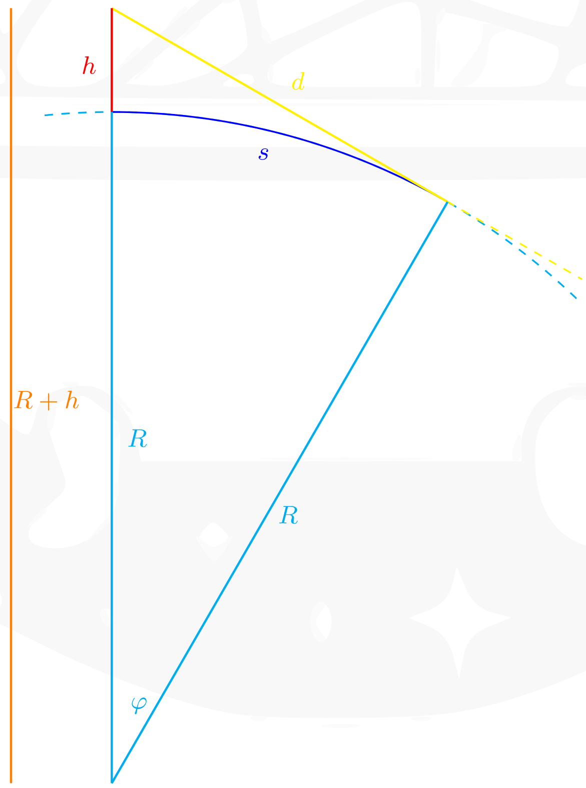

Figure 1: Sketch of the situation.

This sketch shows Earth in cyan, whoose radius is R. The ship is at the top position and its mast has height h (red). The direct line of sight to the horizon (yellow) has length d and the actual distance to the horizon (blue) is s. (I. e. the curved path that follows Earth's surface.)

The exact solution

We are trying to find the length s of the path from the ship's position to the horizon. This is simply:

$$s = R\varphi$$

Notice, that we choose the angle φ to be in radians.

Furthermore, the (yellow) line of sight is orthogonal to the (cyan) radius of Earth at the horizon. Thus, we get a right triangle between Earth's center, the crow's nest and the horizon and can make use of

$$\cos\varphi = \frac{R}{R+h} = \frac{1}{\frac{1}{R}(R+h)} = \frac{1}{1+\frac{h}{R}}.$$

We can define

$$q := \frac{h}{R},$$

so the angle φ only depends on the *ratio* of h and R. With

$$\varphi = \arccos\left(\frac{1}{1+q}\right),$$

we finally obtain the result:

$$\boxed{s = R\arccos\left(\frac{1}{1+q}\right)}$$

Now, with Earth's radius being $R \approx 6371\text{km}$, a crow's nest with a height of $h = 15\text{m}$ yields a q-value of

$$q \approx \frac{15\text{m}}{6371\text{km}} \approx 2.35 \cdot 10^{-6}$$

and a distance to the horizon of

$$s(h = 15\text{m}) \approx 13.8\text{km}.$$

An approximate solution

Well, we found the solution - even an exact one - that we could be happy with. (If we have a calculator.)

However, we might want to be able to do a "quick estimate" in our heads. After all, the question of how far we can see is not limited to a ship with a mast of constant height but we might also go for another hike into the mountains and wonder about the area that is covered by our eyes.

Since we already made some approximating assumptions like Earth being a perfect sphere and neglected additional effects like light bending inside the atmosphere, it would seem reasonable to sacrifice a little more of the precision in order to ease the calculations. (In any case, as a physicist you get used to stuff like that.)

Since we deal with a right triangle, we can use the Pythagorean theorem:

$$\begin{aligned} d^2 + R^2 =& (R+h)^2 \\ =& R^2(1+q)^2 \\ =& R^2(1+2q+q^2) \end{aligned}$$

As long as the angle φ is small, we can approximate:

$$s \approx d$$

Inserting this and subtracting R^2 on both sides of the equation yields:

$$\begin{aligned} s^2 \approx& R^2(2q+q^2) \\ \Rightarrow \quad s \approx& R\sqrt{2q+q^2} \\ =& R\sqrt{q}\sqrt{2+q} \end{aligned}$$

Now, recall that we already had the example of h being 15 meters, which gave $q \sim 10^{-6}$, which is really small compared to 2. (This reflects the fact that the height h is usually dead small compared to Earth's radius. Even on Mount Everest, we would still just get $q \approx 1.389 \cdot 10^{-3} \ll 1$.)

Approximately, we can neglect q in the last square-root-term:

$$\sqrt{2+q} \approx \sqrt{2}$$

This implies

$$\boxed{s \approx \sqrt{2Rh},}$$

or, if we prefer actual numbers:

$$s \approx \sqrt{\frac{2 \cdot 6371}{1000}} \sqrt{\frac{h}{1\text{m}}} \cdot 1\text{km} \approx 3.560\text{km} \cdot \sqrt{\frac{h}{1\text{m}}}$$

In other words:

> Take the square root of your height in *meters* and multiply by 3.560 to get the horizon's distance to yourself in *kilometers*.

Without a doubt, this is a much easier result, but how long will it yield reasonably good approximations to the "real" value?

At least with $h = 15$m, we again get $s \approx 13.8$km, like calculated in the last section.

Comparing exact and approximate solution.

Well, to answer the limit af a "reasonably good approximation" to the real value, we have to start by defining what "reasonably good" actually means.

Arbitrarily, we could say the discrepancy between exact solution s_e and approximate solution s_a shall always be smaller than 10%.

If we do that,

$$\frac{|s_a - s_e|}{s_e} < \frac{1}{10}.$$

As a matter of fact,

$$s_a > s_e$$

for at least our relevant domain of $q < 1$. This could be verified, for example, by considering both to be a function of q and doing a taylor series expansion to compare terms.

Alternatively, we could simply plot the solution graphs and their difference graph to convince ourselves:

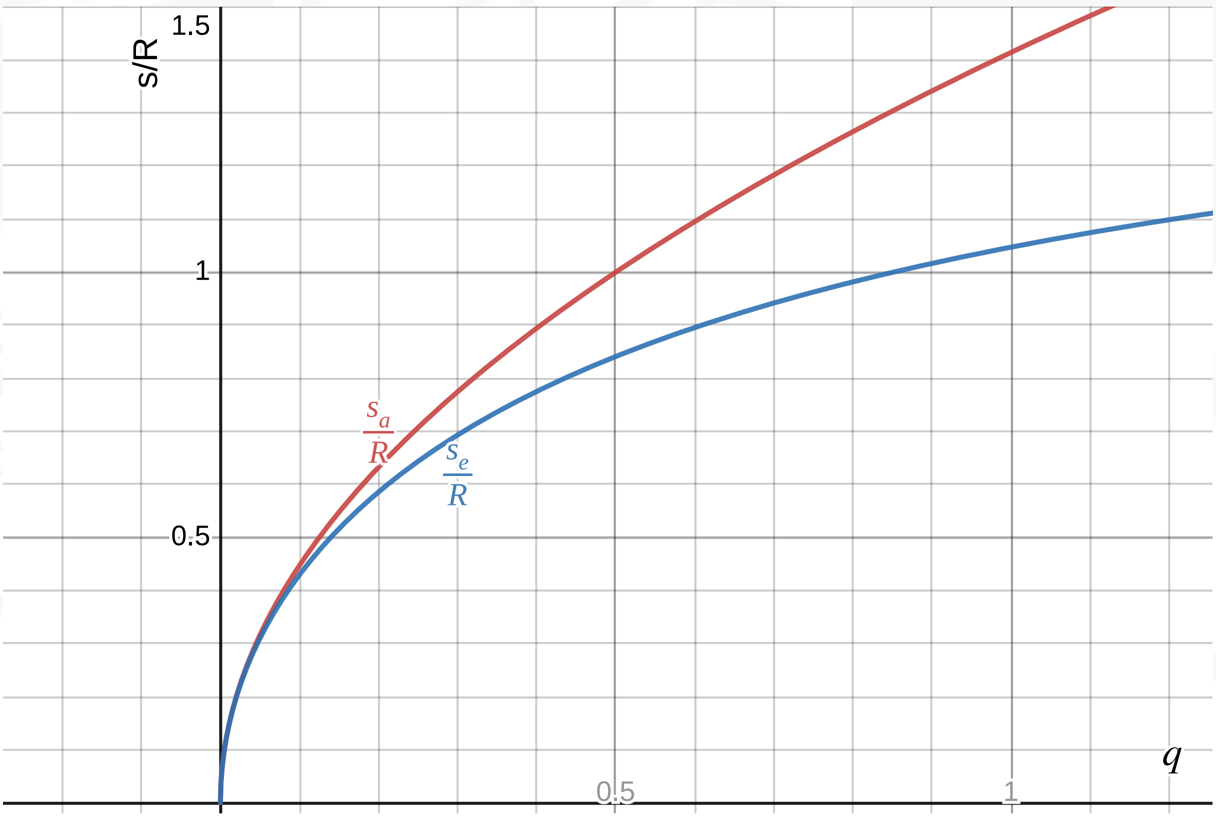

Figure 2: Approximate versus exact solution.

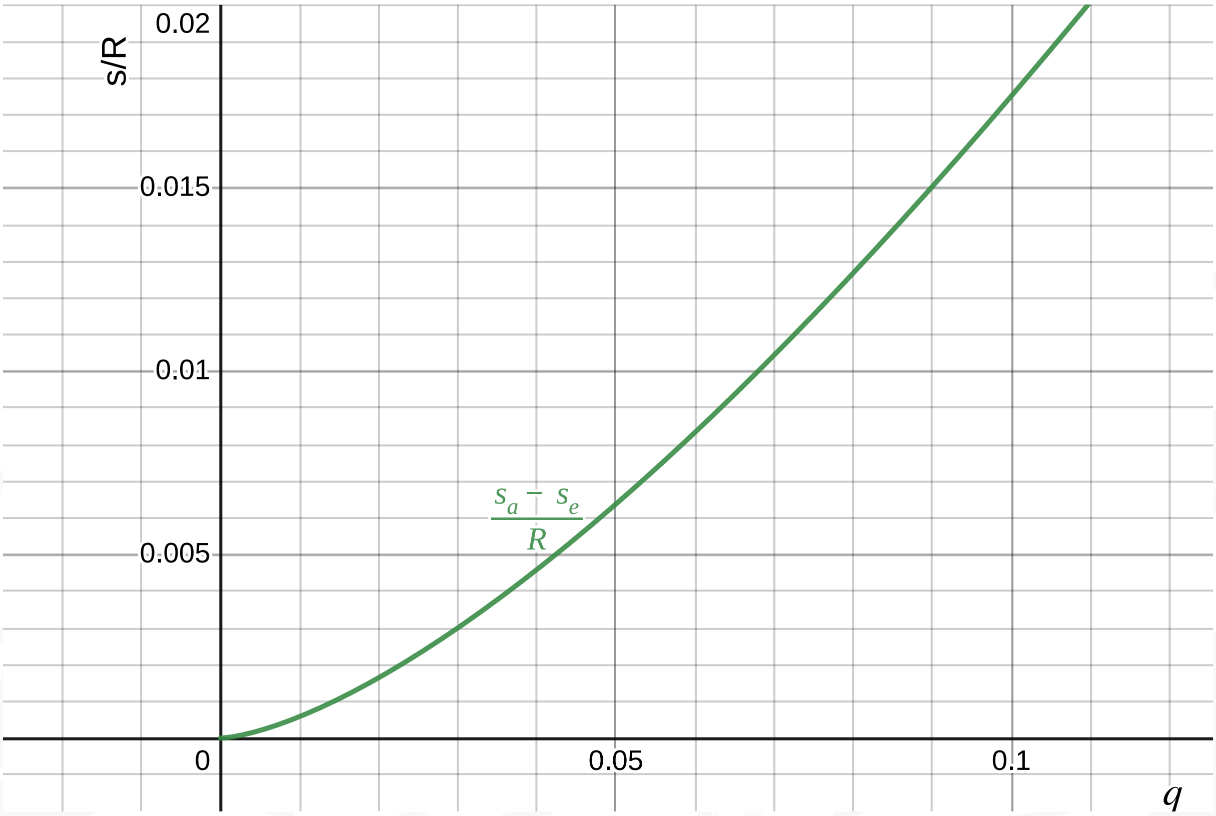

Figure 3: Difference between approximate and exact solution.

Anyways, we wanted to know, how far we can move upwards, before we reach a 10%-error with our approximation. Since we have convinced ourselves, that $s_a > s_e$, we can drop the absolute value bars:

$$\begin{aligned}
\frac{1}{10} &= \frac{s_a - s_e}{s_e} \\
\Leftrightarrow \frac{s_e}{10} &= s_a - s_e \\
\Leftrightarrow \frac{11}{10} s_e &= s_a \\
\Rightarrow \frac{11}{10}\left[R \arccos\left(\frac{1}{1+q_{10}}\right)\right] &= R\sqrt{2q_{10}} \\
\Leftrightarrow \frac{11}{10} \arccos\left(\frac{1}{1+q_{10}}\right) &= \sqrt{2q_{10}}
\end{aligned}$$

This is the end of the rope for analytic calculations in the realm of real numbers.

Using Wolframalpha, we get

$$q_{10} \approx 0.253,$$

which corresponds to

$$h_{10} \approx 1612\text{km}.$$

To contain the error due to our approximation within 10%, we can go up for a little over $\frac{1}{4}$ of Earth's radius!

For comparison, when the ISS is farthest away from Earth, it still only has a height of about 410km. Even the astronauts could use the approximation and the error due to this would be way below 10%!

Now, let us consider a height that might have some practical meaning and assume a plane to be flying at an altitude of 10km over a mountain of height 4000m.

$$\begin{aligned} s_a(h = 10\text{km}) &\approx 356.959\text{km} \\ s_e(h = 10\text{km}) &\approx 356.726\text{km} \\ s_a(h = 4000\text{m}) &\approx 225.760\text{km} \\ s_e(h = 4000\text{m}) &\approx 225.702\text{km} \end{aligned}$$

From the plane, the error is only about 230m, where on the mountain, it is less than 60m!

What can we learn from this?

This example illustrates the might of approximations in physics.

Regularly, approximations are the only thing enabling any kind of calculation. However, it should always be considered, what impact an approximation has on the given problem. (Indeed, a crude approximation might still be improved. This is a job for perturbation theory.)

In any case, it is reasonable to use approximations if the problem at hand simply does not allow to find an exact solution, while also being a matter of circumstances.

Suppose you want to travel a distance of 1000km by car, driving at an average speed of 100kph. In that case, an error of 10% means, that an already exhausting 10-hour-drive might become an 11-hour-drive. However, if you go that distance by plane with an average speed of 800kph, you expect a trip of 1h 15min, which would take 7.5min longer - much less of a bummer.