Arc length in cartesian coordinates

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In physics, we often need to calculate the so-called arc length of a curve or generally use the arc length formula. Therefore, we shall calculate it here for the two-dimensional case.

Definition

Let $y(x)$ be a given function that is continuously differentiable at least once. This means, it could also be a physical trajectory. Furthermore, two points $A(x_a, y(x_a))$ and $B(x_b, y(x_b))$ that are part of the function are given. (see sketch)

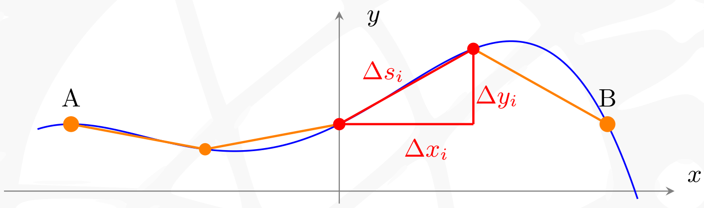

Figure 1: The red and orange polyline's length approximates the length of the curve.

This implies the following question:

How long is the way from A to B along the curve?

Formally, one should define precisely, what the "way" is. For this article, "way" shall be the "physical way", i. e. the length of a string whose shape is $y(x)$. This is the **arc length** $\mathcal{B}_{A,B}$.

The derivation

The line segments Δs_i make a polyline of a length, that can be calculated, given the endpoints. This length is an approximation for the actual arc length $\mathcal{B}_{A,B}$, which gets better, when the Δx_is (and therefore also the Δy_is and Δs_is, respectively) get smaller. A specific segment Δs_i can be expressed using the Pythagorean theorem:

$$\begin{aligned}\Delta s_i =& \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ =& \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i\end{aligned}$$

Chunking the interval $[x_a, x_b]$ into n pieces, the arc length $\mathcal{B}_{A,B}$ is approximated by the polyline of length l_n:

$$\begin{aligned}l_n &= \sum_{i=1}^{n} \Delta s_i \\ &= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i\end{aligned}$$

In the limit $n \to \infty$, the length l_n becomes the actual arc length.

As Δx_i, Δy_i and Δs_i get infinitely small, the sum becomes an integral:

$$\begin{aligned}\mathcal{B}_{A,B} &= \lim_{n\to\infty} l_n \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta s_i(n) \\ &= \int_0^{\mathcal{B}_{A,B}} ds \\ &= \int_{x_A}^{x_B} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx\end{aligned}$$

Thus, the final formula for the arc length is:

$$\boxed{\mathcal{B}_{A,B} = \int_{x_A}^{x_B} \sqrt{1 + (y'(x))^2} dx}$$